PRIME GRAPHS(Finite groups and related topics)

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PRIME GRAPHS

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1. Prime graphs. Let G be a finite group and $\Gamma(G)$ be the prime graph of G. This is the graph such that the vertex-set $V(\Gamma(G)) = \pi(G)$, the set of prime divisors of $|G|$ and two distinct primes p and r are joined by an edge if and only if there exists an element of order pr in G. The concept of prime graph arose from cohomological questions associated with integral representation of finite groups (See Gruenberg[4],[5], Gruenberg-Roggenkamp[6],[7]). Let $n(\Gamma(G))$ be the number of connected components of $\Gamma(G)$ and $d_G(p,r)$ the length of the shortest path between p and r. If there is no path between p and r, then $d_G(p,r)$ is defined to be infinite.

Theorem 1 ([10],[13],[14]).

$$n(\Gamma(G)) = \begin{cases} 1, \\ 2, \\ 3, \\ 4, \\ 5, \\ 6 \end{cases}$$

Theorem 2 ([11]).

$$d_G(p,r) = \begin{cases} 1, \\ 2, \\ 3, \\ 4, \\ \infty \end{cases}$$

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Remark 1. Theorems 1 and 2 hold for any finite group G. The proofs depend upon the classification of finite simple groups. Theorem 1 is the solution of Gruenberg-Kegel's conjecture. We classify not only the number of connected components but also the components themselves. The significance of Theorem 1 can be found in [5],[8],[9],[12] and [15].

Remark 2. If G is solvable or simple, then $d_G(p,r) = 1,2,3$ or $d_G(p,r) = \infty$. For the sporadic simple group G, $d_G(p,r) = 3$ if and only if $G = F_1$ and $p = 29$, $r = 47$ or $G = M_{23}$ and $p = 3$, $r = 7$. Unfortunately we have no application of Theorem 2. We are trying to find applications of Theorem 2.

2. Related topics. Let χ be a character(*resp.* p-Brauer character) of G and L be the set of values of χ on nonidentity elements(*resp.* nonidentity p-regular elements) of G. We say that χ is sharp(*resp.* p-Brauer sharp) if $f_L(\chi(1)) = |G|$ (*resp.* $f_L(\chi(1)) = |G|_{p'}$) where $f_L(x)$ is the monic polynomial of least degree whose set of roots is L. We note that $|G|$ (*resp.* $|G|_{p'}$) always divides $f_L(\chi(1))$ by Blichfeldt's theorem(See [1]). Recently Alvis and Nozawa[1] classified the groups with sharp character χ such that χ takes an irrational value and $(\chi, 1_G) = 1$. Therefore we can assume that L is contained in $\mathbf{Z}$. Let $L = \{l_1, l_2, ..., l_t\}$. The ($p$-Brauer) sharp character χ is said to be t-connected if and only if $L \subseteq \mathbf{Z} - \{\chi(1)-1, \chi(1)+1\}$ and $(\chi(1)-l_i, \chi(1)-l_j) = 1$ for $i \neq j$.

Theorem 3 ([3],[8]). *The following two conditions are equivalent.*

(1) *G has a 2-connected (p-Brauer) sharp character.*

(2) *$\Gamma(G) - \{p\}$ is disconnected.*

Remark 3. $\Gamma(G) - \{p\}$ is a subgraph of $\Gamma(G)$ such that the vertex-set is $V(\Gamma(G)) - \{p\}$. If p does not divide $|G|$, then $\Gamma(G) - \{p\} = \Gamma(G)$ and the result is for ordinary (generalized) characters.

Remark 4. In [1] the authors assume that χ is the character of its representation. However in [3] and [8] χ may not have its representation.

Let $\mathfrak{N}(G)=\{n \in \mathbf{Z} | G \text{ has a conjugacy class } C \text{ with } |C| = n\}$. Thompson made the following conjecture.

Thompson's conjecture. Let G be a finite group and M a non abelian simple group. If $\mathfrak{N}(G)= \mathfrak{N}(M)$ and $Z(G) = 1$, then G is isomorphic with M.

Theorem 4 ([2]). *Thompson's conjecture holds for a finite simple group M with $n(\Gamma(M)) > 1$.*

The proof heavily depends upon the classification of the connected components of prime graphs of finite simple groups in Theorem 1.

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